A Simple Recurrence Within the 3-adics and Mixed-radix 2-adics of the $3x+1$ Accelerated First-Inverse Map

Andrey Rukhin, Ph.D.

Naval Surface Warfare Center — Dahlgren, Dahglren, Virginia, 22448 USA

Abstract

This article analyzes the $3x+1$ *Accelerated First-Inverse map*: we will consider the functional powers of the function $\mathcal{B} : Z_3 \to Z_3$ defined as

$$\mathcal{B}\,(3q+r) = \begin{cases} 3q, & r=0 \\ 4q+1, & r=1 \\ 2q+1, & r=2 \end{cases}$$

where $3q+r \in Z_3$. The values of the functional powers are not the primary focus of these analyses; this paper investigates the properties of their quotients. For any sequence of functional powers $(n_u)_{u\geq 0}$ where $n_u = 3q_u + r_u$, we will study the p-adic expansions and shifts ($p \in \{2,3\}$) of the sequence of quotients $(q_u)_{u\geq 0}$. The main result of this paper is a simple recurrence over the set $\{0,1,2,3\}$ for computing the 3-adic and *mixed-radix* $\mathbf{2}$*-adic* canonical expansions and shifts of the quotients that arise in the orbits of the function $\mathcal{B}$.

Key words: 3x+1 Problem, Collatz Conjecture, Ulam's Conjecture, Kakutani Conjecture, Syracuse Problem, Hailstone Problem, Hensel Codes

1 Introduction

1.1 Background

The scope of this article is the $3x+1$ function, which we describe here:

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Email address: andrey.rukhin@navy.mil (Andrey Rukhin, Ph.D.).

The $3x+1$ **Function:** We denote the classic $3x+1$ function by $\mathcal{C} : \mathbb{Z} \to \mathbb{Z}$, and we define it as

$$\mathcal{C}(x) = \begin{cases} 3x+1 & \text{if } x \underset{2}{\equiv} 1, \\ \frac{x}{2} & \text{if } x \underset{2}{\equiv} 0. \end{cases}$$

The dynamical system arising from the iterates of this function is the well-known $3x+1$ Problem:

The $3x+1$ **Problem**: For all $x \in \mathbb{N}$, does there exist a positive integer n so that $\mathcal{C}^{(n)}(x) = 1$?

The history of this well-studied problem goes back to the problem's formulation by Lothar Collatz in 1937. The author will not attempt to detail the complete background of the problem, but will refer the interested reader to the monographs of Lagarias (Lagarias, 2010) and Wirsching (Wirsching, 1998) for comprehensive summaries that survey the vast landscape of the literature. Over the past decades, authors have considered various properties of the trajectories of $\mathcal{C}$; some examples include its stopping time behavior, the stochastic features of a trajectory, the graphical properties of the $3x+1$ trees, the 2-adic behavior of a trajectory, and properties of trajectories over an extended domain (e.g., the sets $\mathbb{R}$ and $\mathbb{C}$), to name a few.

One research focus pertains to the existence of orbits in the $3x+1$ Problem. In this area, one outstanding open question is whether the function possesses an orbit other than the sequence $(1, 4, 2)$ over the domain of positive integers. This effort will highlight the work of Böhm and Sontacchi (Böhm and Sontacchi, 1978): they demonstrate that the existence of integral orbits is equivalent to the existence of integers that assume a rational expression of the form

$$\frac{\sum_{0 \le u < y} 3^u 2^{x(u)}}{2^{x(y)} - 3^y}$$

where $u < v \implies x(u) < x(v)$, and $x(0) = 0$.

The main focus of this article is the p-adic properties ($p \in \{2, 3\}$) of such expressions that arise from the iterates of the closely related function $\mathcal{B} : Z_3 \to Z_3$, where

$$\mathcal{B}(3k+r) = \begin{cases} 3k & \text{if } r = 0, \\ 4k+1 & \text{if } r = 1, \\ 2k+1 & \text{if } r = 2. \end{cases}$$

We will define this function to be the *Accelerated First-Inverse Map*. The reader can readily connect the function $\mathcal{B}$ with the classic $3x+1$ function over the integers: when $r \in \{1, 2\}$, a trajectory of the classic function starting with

$2^{3-r}k+1$ begins with the terms

$$3(2^{3-r}k+1)+1 \to 2^{2-r}(6k)+4 \to 3k+r \to \cdots{}^{1}.$$

There is a wealth of literature on the inverse $3x+1$ mapping (which is discussed in the unpublished survey of Chamberland[2]); the works of Wirsching (Wirsching, 1994) and Applegate and Lagarias (Applegate and Lagarias, 1995) have pioneered the study of its 3-adic structure, and the work herein is also in this vein.

However, most (if not all!) authors focus on the properties of the values of the iterates. This article will focus on the p-adic properties ($p \in \{2,3\}$) of the quotients of the iterates of the function $\mathcal{B}$.

1.2 Article Summary

Let τ be a positive integer, and let $(r_0, \ldots, r_{\tau-1})$ be a sequence of length τ where $r_w \in \{1,2\}$ for each $w \in \{0, \ldots, \tau-1\}$. We will define such a sequence to be an *admissible (remainder) factor* of length τ. We will also define the infinite, periodic sequence $(r_w)_{w \geq 0}$ where $r_w = r_v$ if $w \underset{\tau}{\equiv} v$. Furthermore, for each $w \geq 0$, we will define $e_w = 3 - r_w$, and we will write $E_w^{(v)} = \sum_{0 \leq u < w} e_{u+v}$ for each $v \geq 0$.

We will analyze the unique element $n \in Z_3$ that satisfies the condition

$$\mathcal{B}^{(v)}(n) \underset{3}{\equiv} r_v$$

for each $v \geq 0$. As demonstrated by Böhm and Sontacchi, this element n (in our notation) equals

$$\frac{\sum_{0 \leq w < \tau} 3^w 2^{E_\tau^{(0)} - E_{w+1}^{(0)}}}{2^{E_\tau^{(0)}} - 3^\tau}.$$

If we write the v-th iterate of $\mathcal{B}$ via the equality $\mathcal{B}^{(v)}(n) = 3k_0^{(v)} + r_v$, then we will present a novel recurrence for generating both the 3-adic digits and the *mixed-radix* **2**-*adic digits* of the quotient $k_0^{(v)}$. We will briefly summarize this recurrence as follows: , For each $w \in \{0, \ldots \tau-1\}$, define $d_{-1}^{(w)}(r_w) = r_w$, and

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[1] We exclude the case where $r = 0$ as the equation

$$3q = \frac{3x+1}{2^a}$$

absurdly implies that $3 \mid 1$ when $a, q, x \in \mathbb{Z}$.

[2] *"An Update on the 3x+1 Problem"*, www.math.grinnell.edu/∼chamberl/papers/3x_survey_eng.pdf

define $b_{-1}^{(w)}(r_w) = 1$. For $u \geq 0$, define

$$s_u^{(w)} = d_{u-1}^{(w+1)} \left(r_{w+1}, \ldots, r_{w+1+u}\right) - b_{u-1}^{(w)} \left(r_w, \ldots, r_{w+u}\right),$$

define $d_u^{(w)} \left(r_w, \ldots, r_{w+1+u}\right)$ to equal

$$\left[2^{e_w}\right]^{-1} s_u^{(w)} \bmod 3,$$

and define $b_u^{(w)} \left(r_w, \ldots, r_{w+1+u}\right)$ to equal

$$[-3]^{-1} s_u^{(w)} \bmod 2^{e_w}.$$

We will show that the 3-adic canonical expansion of $k_0^{(v)}$ equals $d_0^{(v)} d_1^{(v)} \cdots$ where

$$d_u^{(v)} = d_u^{(v)} \left(r_v, \ldots, r_{v+u+1}\right)$$

for each $u \geq 0$.

Furthermore, define the term $k_{u+1}^{(v)}$ to be the 3-adic shift of $k_u^{(v)}$: assume that the equality $k_u^{(v)} = 3k_{u+1}^{(v)} + d_u^{(v)}$ holds for $u \geq 0$. We will demonstrate the equality

$$k_u^{(v)} = \frac{\sum_{0 \leq w < \tau} 3^w 2^{E_\tau^{(v)} - E_{w+1}^{(v)}} s_u^{(w+v)}}{2^{E_\tau^{(v)}} - 3^\tau}.$$

We will also introduce a *mixed-radix* **2**-*adic canonical expansion* of $k_0^{(v)}$: this expansion will include the $b_u^{(w)}$ terms decided by the above recurrence. Finally, we will introduce the *mixed-radix* **2**-*adic shift* of $k_0^{(v)}$ in terms of the shifts of the τ quotients $k_0^{(0)}, \ldots, k_0^{(\tau-1)}$.

1.3 Analyses of the Iterate Quotients

We will first prescribe a iterate remainder sequence $(r_u)_{u \geq 0}$ where each $r_u \in \{1, 2\}$. We will study certain features of an iteration sequence $(n_u)_{u \geq 0}$, where

$$n_{u+1} = \mathcal{B}\left(n_u\right),$$

and the iterate n_u in the sequence satisfies the equivalence

$$n_u \underset{3}{\equiv} r_u$$

for all u ($u \geq 0$). We will define the sequence $(r_u)_{u \geq 0}$ to be the *(iterate) remainder trajectory*, and we will say that this remainder trajectory generates the *iterate trajectory* $(n_u)_{u \geq 0}$.

First, we will analyze such iterate trajectories when the remainder trajectory is finite. If we write $n_u = 3k_u + r_u$ for each $u \geq 0$, we will show that a prescribed remainder trajectory of length $u+1$ determines the first u 3-adic digits of the the canonical expansion of the initial quotient k_0.

We will diverge from previous analyses by expressing the u-th iterate n_u in a different manner. As before, we define $e_u = 3 - r_u$ for all $u \geq 0$. Assuming that our trajectory begins with n_0 where $n_0 = 3k_0 + r_0$, we will write

$$n_1 = \mathcal{B}(n_0) = 2^{e_0} k_0 + 1.$$

We will write $k_0 = 3k_1 + d_0$ where the value $d_0 \in \{0, 1, 2\}$. The condition $n_1 \underset{3}{\equiv} r_1$ yields the equality

$$2^{e_0} k_0 + 1 = 3\left(2^{e_0} k_1 + \frac{2^{e_0} d_0 - (r_1 - 1)}{3}\right) + r_1;$$

here, we assume that the expression $\frac{2^{3-r_0} d_0 - (r_1 - 1)}{3}$ is integer-valued. This integrality condition decides a unique residue $d_0 = d_0(r_0, r_1)$: this residue satisfies the equivalence

$$d_0 \underset{3}{\equiv} 2^{3-r_0}(r_1 - 1).$$

We will now proceed inductively: assume that we can write

$$n_u = 3\left(2^{E_u} k_u + P_u\right) + r_u$$

where

i. the term k_u satisfies the equality $k_{u-1} = 3k_u + d_{u-1}$ ($d_{u-1} \in \{0, 1, 2\}$);
ii. the term $d_{u-1} = d_{u-1}(r_0 \ldots, r_u)$;
iii. the terms P_u and E_u are integers;
iv. the term $P_u = P_u(r_0, \ldots, r_u)$.

The next iterate n_{u+1} equals

$$2^{e_u}\left(2^{E_u} k_u + P_u\right) + 1;$$

we will write $k_u = 3k_{u+1} + d_u$ ($d_u \in \{0, 1, 2\}$), we will write $E_{u+1} = e_u + E_u$, and we will express the iterate n_{u+1} as

$$3\left(2^{E_{u+1}} k_{u+1} + P_{u+1}\right) + r_{u+1}$$

where

$$P_{u+1} = \frac{2^{E_{u+1}} d_u + 2^{e_u} P_u - (r_{u+1} - 1)}{3}.$$

We require that the addend P_{u+1} is an integer; thus, we require that the 3-adic digit d_u satisfies the equivalence

$$d_u \underset{3}{\equiv} -2^{E_{u+1}} \left[2^{e_u} P_u - (r_{u+1} - 1)\right].$$

Consequently, we can write $d_u = d_u(r_0, \ldots, r_{u+1})$ and $P_{u+1} = P_{u+1}(r_0, \ldots, r_{u+1})$.

From the recurrence

$$k_u = 3k_{u+1} + d_u \ (u \geq 0),$$

we can conclude that the starting quotient k_0 admits a 3-adic canonical expansion with the mantissa $(d_u)_{0 \leq w \leq u}$.

We will study the various terms arising from this formulation of the iterates; we will now define the terms within the expression

$$n_u = 3\left(2^{E_u} k_u + P_u\right) + r_u$$

of the u-th iterate as follows.

Definition 1. Let

$$n_u = \mathcal{B}^{(u)}(3k_0 + r_0) = 3\left(2^{E_u} k_u + P_u\right) + r_u$$

be the u-th iterate of $\mathcal{B}(3k_0 + r_0)$. We define

i. the remainder term r_u ($r_u \in \{1, 2\}$) is the u-th *iterate remainder*;
ii. the term e_u, $e_u = 3 - r_u$, is the u-th *exponent*;
iii. the term E_u, $E_u = \sum_{0 \leq w < u} e_w$, is the u-th *exponent sum*;
iv. the expression $2^{E_u} k_u + P_u$ is the u-th *quotient*;
v. the term k_u is the u-th 3-*adic shift* of k_0 where

$$k_u = 3k_{u+1} + d_u,$$

and
vi. the term d_u, ($d_u \in \{0, 1, 2\}$) is the u-th *(ternary) quotient remainder*;
vii. the term P_u, the u-th *(ternary) prefix addend*, is integer-valued, and it is defined recursively as

$$P_u = \begin{cases} 0 & u = 0, \\ \frac{2^{E_u} d_{u-1} + 2^{e_{u-1}} P_{u-1} - (r_u - 1)}{3} & u > 0. \end{cases}$$

1.4 Notation Summary

If a, b and q are integers with $q \neq 0$, we will say that $a \underset{q}{\equiv} b$ if and only if a is equivalent to b modulo q; otherwise, we will write $a \underset{q}{\not\equiv} b$. We will express the

set $\{1, \ldots, i\}$ of the first i positive integers as $[i]$. We will write $[i)$ to denote the set $[i-1]$. We will also write $[i]_0$ and $[i)_0$ to denote the sets $[i] \cup \{0\}$ and $[i) \cup \{0\}$, respectively.

Let $\mathbf{x} = (x_0, \ldots, x_{z-1})$ be a list of length z over $\{1, 2\}$. We will use the standard notation $\mathbf{x}^u$ to denote the u-fold concatenation

$$\underbrace{(x_0, \ldots, x_{z-1}, \ldots, x_0, \ldots, x_{z-1})}_{u \text{ times}}.$$

We will also define the infinite list $\mathbf{x}^\infty$ to be the periodic list $(x_u)_{u \geq 0}$ where $x_u = x_v$ if $u \underset{z}{\equiv} v$. We will say that the list $\mathbf{x}$ is the *factor* of the list $\mathbf{x}^u$ for both finite and infinite u. Context permitting, we will write the factor x as $\mathbf{x}_0 \cdots \mathbf{x}_{z-1}$.

Let $v \in \mathbb{Z}$. The *shift* of the list $\mathbf{x}$ (of index v), denoted by $\mathbf{x}^{(v)}$, is the list $(x_0^{(v)}, \ldots, x_{z-1}^{(v)})$, where $x_c^{(v)} = x_{s(c)}$, and

$$s(c) \underset{z}{\equiv} c + v$$

for all $c \in [z)_0$. When $v = 0$, we may naturally omit the superscript. We will denote the set of the z cyclic shifts of $\mathbf{x}$ by

$$\mathcal{O}(\mathbf{x}) = \left\{ \mathbf{x}^{(v)} \mid v \in [z)_0 \right\}.$$

The shift (of index v) of the infinite list $\mathbf{x}^\infty$ is the list $\left[\mathbf{x}^{(v)}\right]^\infty$; similar to the case of finite lists, we will write $\mathcal{O}(\mathbf{x}^\infty) = \left\{ [\mathbf{x}^{(v)}]^\infty \mid v \in [z)_0 \right\}$.

Assume g is a function where $g : \{1, 2\}^z \to \mathbb{Q}$; we will adopt the above notational convention and write $g^{(v)}(\mathbf{x}) = g(\mathbf{x}^{(v)})$. We will also apply this convention when handling infinite trajectories. Let $(x_u)_{u \geq 0}$ denote the infinite, periodic trajectory $\mathbf{x}^\infty$. Let y be a function where $y : \mathbb{N}_0 \to \mathbb{N}$, and let

$$g_u : \{1, 2\}^{y(u)} \to \mathbb{Q}$$

be a function that is evaluated on the prefix list $(x_w)_{0 \leq w < y(u)}$. For $v \in \mathbb{Z}$, we will write $g_u^{(v)}$ to denote the function value of g_u evaluated on the prefix list $(x_w^{(v)})_{0 \leq w < y(u)}$ of $[\mathbf{x}^{(v)}]^\infty$.

See Table 1 for an illustration of this notation applied to the orbit $(-5, -7)$ within the Accelerated First-Inverse Map; this orbit corresponds to the orbit

$$-5 \to -14 \to -7 \to -20 \to -10 \to -5 \to \cdots$$

within the classic $3x + 1$ map (over the domain of all integers).

$\mathbf{r}^{(0)} = \mathbf{12}$

u	$r_u^{(0)}$	$d_{u-1}^{(0)}$	$E_u^{(0)}$	$P_u^{(0)}$
0	1		0	0
1	2	1	2	1
2	1	2	3	6
3	2	2	5	29
4	1	2	6	62
5	2	2	8	253

(a) $(\mathbf{12})^3$

$\mathbf{r}^{(1)} = \mathbf{21}$

u	$r_u^{(1)}$	$d_{u-1}^{(1)}$	$E_u^{(1)}$	$P_u^{(1)}$
0	2		0	0
1	1	0	1	0
2	2	2	3	5
3	1	2	4	14
4	2	2	6	61
5	1	2	7	126

(b) $(\mathbf{21})^3$

Table 1
The first several terms of the trajectories of the set of cyclic shifts $\mathcal{O}(\mathbf{12})$ are tabulated. For the remainder trajectory $(\mathbf{12})^\infty$, the 3-adic representation of the initial quotient is $122\cdots$ (which encodes the value -2); for the remainder trajectory $(\mathbf{21})^\infty$, the 3-adic representation of the initial quotient is $022\cdots$ (which encodes the value -3). However, for almost all values of u, the prefix addend $P_u^{(0)} \underset{3^u}{\not\equiv} 3^u - 2$, and the prefix addend $P_u^{(1)} \underset{3^u}{\not\equiv} 3^u - 3$ (highlighting the difference between the prefix addends and the residue classes of $-2 \bmod 3^u$ and $-3 \bmod 3^u$, respectively).
The corresponding initial iterate values are -5 and -7, respectively. Within the classic $3x + 1$ map, the value $\mathcal{C}(-5) \rightarrow \frac{-15+1}{2} = -7$, and the value $\mathcal{C}(-7) \rightarrow \frac{-21+1}{4} = -5$.

We will proceed to the next section; it includes an intermediate lemma that will assist us in demonstrating the results within this article. Afterwards, the main theorems of this article are presented and proven, and we conclude the article with a brief discussion on a variety of potential directions for future research.

2 Theory/Calculation

This section begins with a lemma that is required for the main results of this article.

Lemma 2 (Diagonal Differences). *Let τ be a positive integer, and let $\mathbf{r}$ be an admissible factor of length τ where $\mathbf{r} = (r_0, \ldots, r_{\tau-1})$. For each $w \in [\tau)_0$, let $\left(d_u^{(w)}\right)_{u\geq 0}$, $\left(P_u^{(w)}\right)_{u\geq 0}$, and $\left(E_u^{(w)}\right)_{u\geq 0}$ be the quotient remainder trajectory, prefix addend trajectory, and exponent sum trajectory generated by $\left[\mathbf{r}^{(w)}\right]^\infty$, respectively.*

For all $u \geq 0$, the prefix addend difference

$$P_{u+1}^{(w)}\left(r_0^{(w)},\ldots,r_{u+1}^{(w)}\right) - P_u^{(w+1)}\left(r_0^{(w+1)},\ldots,r_u^{(w+1)}\right)$$

equals

$$2^{E_u^{(w+1)}} b_u^{(w)}\left(r_0^{(w)},\ldots,r_{u+1}^{(w)}\right)$$

where the term $b_u^{(w)}\left(r_0^{(w)},\ldots,r_{u+1}^{(w)}\right) \in \left[2^{e_0^{(w)}}\right)_0$*; furthermore, for $u > 0$, the term $b_u^{(w)}$ admits the recurrence*

$$b_u^{(w)} = \frac{2^{e_0^{(w)}} d_u^{(w)} - d_{u-1}^{(w+1)} + b_{u-1}^{(w)}}{3}$$

with the the initial condition $b_0^{(w)} = r_1^{(w)} - 1$.

PROOF.

Assume the hypotheses and notation within the statement of the lemma. We will demonstrate the claim by induction on u.

For all choices of $\left(r_0^{(w)}, r_1^{(w)}\right) \in \{1,2\}^2$, the difference

$$P_1^{(w)}\left(r_0^{(w)}, r_1^{(w)}\right) - P_0^{(w+1)}\left(r_1^{(w)}\right) = P_1^{(w)}\left(r_0^{(w)}, r_1^{(w)}\right) = 2^0(r_1^{(w)} - 1);$$

if the expression

$$\frac{2^{3-r_0^{(w)}} d - (r_1^{(w)} - 1)}{3}$$

is to be integer-valued, and if $d \in \{0,1,2\}$, then

$$P_1^{(w)}\left(r_0^{(w)}, r_1^{(w)}\right) = r_1^{(w)} - 1 \in \{0,1\}.$$

We will now bring our attention to the difference of the prefix addend terms

$$P_{u+1}^{(w)}\left(r_0^{(w)},\ldots,r_{u+1}^{(w)}\right) - P_u^{(w+1)}\left(r_0^{(w+1)},\ldots,r_u^{(w+1)}\right) \tag{1}$$

when $u > 0$. The first term in this difference equals

$$\frac{2^{E_{u+1}^{(w)}} d_u^{(w)} - 2^{e_u^{(w)}} P_u^{(w)} - (r_{u+1}^{(w)} - 1)}{3};$$

the second term equals

$$\frac{2^{E_u^{(w+1)}} d_{u-1}^{(w+1)} - 2^{e_{u-1}^{(w+1)}} P_{u-1}^{(w+1)} - (r_u^{(w+1)} - 1)}{3}.$$

We can write $E_{u+1}^{(w)} = e_0^{(w)} + E_u^{(w+1)}$, we can write $e_{u-1}^{(w+1)} = e_u^{(w)}$, and we can express the difference in (1) as

$$\frac{2^{E_u^{(w+1)}}\left[2^{e_0^{(w)}} d_u^{(w)} - d_{u-1}^{(w+1)}\right] - 2^{e_u^{(w)}}\left(P_u^{(w)} - P_{u-1}^{(w+1)}\right)}{3}. \tag{2}$$

By induction, the difference

$$P_u^{(w)} - P_{u-1}^{(w+1)} = 2^{E_{u-1}^{(w+1)}} b_{u-1}^{(w)},$$

where

$$b_{u-1}^{(w)} = b_{u-1}^{(w)}\left(r_0^{(w)}, \ldots, r_u^{(w)}\right) \in \left[2^{e_0^{(w)}}\right)_0.$$

We can complete the inductive argument by writing $E_u^{(w+1)} = E_{u-1}^{(w+1)} + e_u^{(w)}$, and rewriting (2) as

$$2^{E_u^{(w+1)}}\left[\frac{2^{e_0^{(w)}} d_u^{(w)} - d_{u-1}^{(w+1)} + b_{u-1}^{(w)}}{3}\right].$$

Define

$$b_u^{(w)} = b_u^{(w)}\left(r_0^{(w)}, \ldots, r_{u+1}^{(w)}\right) = \frac{2^{e_0^{(w)}} d_u^{(w)} - d_{u-1}^{(w+1)} + b_{u-1}^{(w)}}{3};$$

this term must be an integer as the expression in (2) is a difference of two integers. Furthermore, the terms $d_u^{(w)}, d_{u-1}^{(w+1)} \in [3)_0$, and, by induction, the term $b_{u-1}^{(w)} \in \left[2^{e_0^{(w)}}\right)_0$. Thus, we can readily bound this expression with the inequalities

$$-\frac{2}{3} \leq b_u^{(w)} \leq \frac{11}{3}$$

when $r_0^{(w)} = 1$, and with inequalities

$$-\frac{2}{3} \leq b_u^{(w)} \leq \frac{5}{3}$$

when $r_0^{(w)} = 2$.

As the term $b_u^{(w)} \in \mathbb{Z}$, we have

$$b_u^{(w)} \in \left[2^{e_0^{(w)}}\right)_0.$$

□

We will define the sequence $\left(b_u^{(w)}\right)_{u \geq 0}$ to be the *diagonal difference trajectory* generated by $\left[\mathbf{r}^{(w)}\right]^{\infty}$.

We will describe the role of the diagonal difference trajectories in a novel representation of the quotient. To this end, we will introduce the ring of *mixed-radix* $\mathbf{m}$*-adic integers*.

2.1 The Ring of Mixed-Radix $\mathbf{m}$*-adic Integers*

Our construction of the ring of mixed-radix $\mathbf{m}$-adic integers is analogous to the algebraic construction of m-adics (see Sutherland (2013)).

Let m be a positive integer where $m \geq 2$, and let τ be a positive integer. Let $\mathbf{g}$ be a list of positive integers of length τ where $\mathbf{g} = (g_0, \ldots, g_{\tau-1})$, and let $(g_u)_{u \geq 0}$ denote the infinite, periodic sequence $\mathbf{g}^\infty$. For each $u \geq 0$, define G_u to be the prefix sum $\sum_{0 \leq v < u} g_v$.

Let R_u denote the ring $\mathbb{Z} \backslash m^{G_u}\mathbb{Z}$ for each $u \geq 1$. For each element r of R_u, we assume the mixed-radix representation

$$r = \sum_{0 \leq v < u} \alpha_v m^{G_v}$$

where $\alpha_v \in [m^{g_v})_0$.

Define the morphism $f_u : R_{u+1} \to R_u$ to be reduction modulo m^{G_u}. We define the ring of *(mixed-radix)* $\mathbf{m}$*-adic integers*[3] $Z_{\mathbf{m},\mathbf{g}}$ to be the inverse limit

$$Z_{\mathbf{m},\mathbf{g}} = \varprojlim R_u$$

of the inverse system $(R_u, f_u)_{u \geq 1}$. We define the sequence $\mathbf{g}^\infty$ to be the *gradation sequence* of $Z_{\mathbf{m},\mathbf{g}}$ (and the factor $\mathbf{g}$ to be the *gradation factor* of $Z_{\mathbf{m},\mathbf{g}}$).

Context permitting, we will omit the gradation factor from the subscript.

We will now proceed to the main results of this article.

3 Main Results

3.1 A Recurrence on the Ternary Quotient Remainders and the Diagonal Differences

Let $\mathbf{r}$ be an admissible remainder factor of length τ. Our first result, albeit a straightforward consequence of Lemma 2, provides a recursive construction of

[3] We will write $\mathbf{m}$ instead of m to emphasize the mixed-radix nature of the ring.

the 3-adic digits and the diagonal difference terms generated by the τ shifts of $\mathbf{r}^\infty$.

Theorem 3. *Let τ be a positive integer, and let $\mathbf{r}$ be an admissible factor of length τ where $\mathbf{r} = (r_0, \ldots, r_{\tau-1})$. For each $w \in [\tau)_0$, the ternary quotient remainders $(d_u^{(w)})_{u \geq 0}$ and the diagonal difference terms $(b_u^{(w)})_{u \geq 0}$ generated by $[\mathbf{r}^{(w)}]^\infty$ satisfy the following recurrence: for $u \geq 0$, the quotient remainder*

$$d_u^{(w)} = d\left(r_0^{(w)}, \ldots, r_{u+1}^{(w)}\right) = \left[2^{e_0^{(w)}}\right]^{-1} \left[d\left(r_1^{(w)}, \ldots, r_{u+1}^{(w)}\right) - b\left(r_0^{(w)}, \ldots, r_u^{(w)}\right)\right] \bmod 3,$$

and the diagonal difference term

$$b_u^{(w)} = b\left(r_0^{(w)}, \ldots, r_{u+1}^{(w)}\right) = [-3]^{-1} \left[d\left(r_1^{(w)}, \ldots, r_{u+1}^{(w)}\right) - b\left(r_0^{(w)}, \ldots, r_u^{(w)}\right)\right] \bmod 2^{e_0^{(w)}},$$

along with the initial conditions $d_{-1}\left(r_0^{(w)}\right) := r_0^{(w)}$ and $b_{-1}\left(r_0^{(w)}\right) := 1$.

PROOF. Assume the hypotheses and notation within the statement of the theorem.

Fix $w \in [\tau)_0$. Consider the following equation from Lemma 2: for $u > 0$, the diagonal difference term

$$b_u^{(w)} = \frac{2^{e_0^{(w)}} d_u^{(w)} + b_{u-1}^{(w)} - d_{u-1}^{(w+1)}}{3}.$$

If we rewrite this equation as

$$2^{e_0^{(w)}} d_u^{(w)} = 3b_u^{(w)} + d_{u-1}^{(w+1)} - b_{u-1}^{(w)},$$

then we arrive at the equivalence relation

$$d_u^{(w)} \underset{3}{\equiv} \left[2^{e_0^{(w)}}\right]^{-1} \left[d_{u-1}^{(w+1)} - b_{u-1}^{(w)}\right].$$

If we rewrite this equation as

$$3b_u^{(w)} = 2^{e_0^{(w)}} d_u^{(w)} + b_{u-1}^{(w)} - d_{u-1}^{(w+1)},$$

then we arrive at the equivalence relation

$$b_u^{(w)} \underset{2^{e_0^{(w)}}}{\equiv} [-3]^{-1} \left[d_{u-1}^{(w+1)} - b_{u-1}^{(w)}\right].$$

The stated initial conditions satisfy the equivalences

$$d_0^{(w)}(r_0^{(w)}, r_0^{(w+1)}) \underset{3}{\equiv} \left[2^{e_0^{(w)}}\right]^{-1} \left[r_0^{(w+1)} - 1\right]$$

and

$$b_0^{(w)}(r_0^{(w)}, r_0^{(w+1)}) \underset{2^{e_0^{(w)}}}{\equiv} [-3]^{-1} \left[r_0^{(w+1)} - 1\right].$$

□

For each $u \geq 0$, and for each $w \in [\tau)_0$, we will now write $s_u^{(w)}$ to express the difference

$$d_{u-1}^{(w+1)} - b_{u-1}^{(w)}.$$

3.2 The 3-adic Shifts of the Iterates

We will proceed by bridging our results with the analyses of the periodic iterate trajectory $\left(n_0^{(v)}\right)_{v\geq 0}$ in Böhm and Sontacchi (1978); they have shown that the iterate value $n_0^{(v)}$ ($n_0^{(v)} = 3k_0^{(v)} + r_0^{(v)}$) generated by $\left[\mathbf{r}^{(v)}\right]^\infty$ equals

$$\frac{\sum_{0\leq w<\tau} 3^w 2^{E_\tau^{(v)} - E_{w+1}^{(v)}}}{2^{E_\tau^{(v)}} - 3^\tau}.$$

We will now incorporate the result of Theorem 3 to establish a recurrence for computing the 3-adic shifts of $n_0^{(v)}$.

Theorem 4. *Let τ be a positive integer, and let $\mathbf{r}$ be an admissible factor of length τ where $\mathbf{r} = (r_0, \ldots, r_{\tau-1})$. Let v be an integer, and let $n_0^{(v)}$ be the initial iterate decided by $\left[\mathbf{r}^{(v)}\right]^\infty$ where*

$$n_0^{(v)} = \frac{\sum_{0\leq w<\tau} 3^w 2^{E_\tau^{(v)} - E_{w+1}^{(v)}}}{2^{E_\tau^{(v)}} - 3^\tau}.$$

If we write $n_0^{(v)} = 3k_0^{(v)} + r_0^{(v)}$, and if we write $k_u^{(v)} = 3k_{u+1}^{(v)} + d_u^{(v)}$ for $u \geq 0$, then

$$k_u^{(v)} = \frac{\sum_{0\leq w<\tau} 3^w 2^{E_\tau^{(v)} - E_{w+1}^{(v)}} s_u^{(v+w)}}{2^{E_\tau^{(v)}} - 3^\tau}$$

for $u \geq 0$.

PROOF.

Assume the hypotheses and notation in the statement of the theorem. We will prove the claim by induction on u.

Let D denote the non-zero quantity $2^{E_\tau^{(v)}} - 3^\tau$. We will begin by demonstrating the equality

$$D\left(3k_0^{(v)} + r_0^{(v)}\right) = 3\sum_{0\le w<\tau} 3^w 2^{E_\tau^{(v)}-E_{w+1}^{(v)}}[r_1^{(v+w)} - 1] + Dr_0^{(v)};$$

solving for $k_0^{(v)}$ yields the first desired equality in the theorem statement. This step of the proof makes use of the identity

$$2^{e_w^{(v)}} r_w^{(v)} = 2^2(1) = 2^1(2) = 4$$

for all $w \in [\tau)_0$. The following chain of equalities hold:

$$\begin{aligned}
D\left(3k_0^{(v)} + r_0^{(v)}\right) &= \sum_{0\le w<\tau} 3^w 2^{E_\tau^{(v)}-E_{w+1}^{(v)}} \\
&= \sum_{0\le w<\tau} 3^w 2^{E_\tau^{(v)}-E_{w+1}^{(v)}} \left(2^{e_w^{(v)}} r_w^{(v)}\right) - 3\sum_{0\le w<\tau} 3^w 2^{E_\tau^{(v)}-E_{w+1}^{(v)}} \\
&= \left[2^{E_\tau^{(v)}} r_0^{(v)} + \sum_{1\le w<\tau+1} 3^w 2^{E_\tau^{(v)}-E_w^{(v)}} r_1^{(v+w-1)} - 3^\tau r_0^{(v)}\right] - 3\sum_{0\le w<\tau} 3^w 2^{E_\tau^{(v)}-E_{w+1}^{(v)}} \\
&= \sum_{0\le w<\tau} 3^{w+1} 2^{E_\tau^{(v)}-E_{w+1}^{(v)}} r_1^{(v+w)} - \sum_{0\le w<\tau} 3^{w+1} 2^{E_\tau^{(v)}-E_{w+1}^{(v)}} + r_0^{(v)}(2^{E_\tau^{(v)}} - 3^\tau) \\
&= 3\sum_{0\le w<\tau} 3^w 2^{E_\tau^{(v)}-E_{w+1}^{(v)}}[r_1^{(v+w)} - 1] + Dr_0^{(v)},
\end{aligned}$$

as desired.

Assume the claim for $u \ge 0$. The inductive step of the proof makes use of the identity

$$2^{e_0^{(c)}} d_u^{(c)} - 3b_u^{(c)} = d_{u-1}^{(c+1)} - b_{u-1}^{(c)} = s_u^{(c)} \tag{3}$$

for all $c \in \mathbb{Z}$; this is an immediate consequence of the Lemma 2. We will write

$$\begin{aligned}
D\left(3k_{u+1}^{(v)} + d_u^{(v)}\right) &= Dk_u^{(v)} \\
&= \sum_{0 \le w < \tau} 3^w 2^{E_\tau^{(v)} - E_{w+1}^{(v)}} s_u^{(v+w)} \\
&= \sum_{0 \le w < \tau} 3^w 2^{E_\tau^{(v)} - E_{w+1}^{(v)}} \left[d_{u-1}^{(v+w+1)} - b_{u-1}^{(v+w)}\right] \\
&= \sum_{0 \le w < \tau} 3^w 2^{E_\tau^{(v)} - E_{w+1}^{(v)}} \left[2^{e_w^{(v)}} d_u^{(v+w)} - 3b_u^{(v+w)}\right] \\
&= \left[2^{E_\tau^{(v)}} d_u^{(v)} + \sum_{1 \le w < \tau+1} 3^w 2^{E_\tau^{(v)} - E_w^{(v)}} d_u^{(v+w)} - 3^\tau d_u^{(v+\tau)}\right] - 3 \sum_{0 \le w < \tau} 3^w 2^{E_\tau^{(v)} - E_{w+1}^{(v)}} b_u^{(v+w)} \\
&= \sum_{1 \le w < \tau+1} 3^w 2^{E_\tau^{(v)} - E_w^{(v)}} \left[d_u^{(v+w)} - b_u^{(v+w-1)}\right] + Dd_u^{(v)} \\
&= 3 \sum_{0 \le w < \tau} 3^w 2^{E_\tau^{(v)} - E_{v+w+1}^{(v)}} s_{u+1}^{(v+w)} + Dd_u^{(v)};
\end{aligned}$$

solving for $k_{u+1}^{(v)}$ completes the inductive argument.

□

3.3 The **2***-adic Shifts of the Iterates*

Define

$$\mathbf{g}^{(v)} = \left(3 - r_{\tau-1}^{(v)}, \ldots, 3 - r_0^{(v)}\right).$$

for $v \in \mathbb{Z}$.

We will now incorporate the result of Theorem 3 to establish a recurrence for computing the **2**-adic shifts (with gradation factor $\mathbf{g}^{(v)}$) of $n_0^{(v)}$.

Theorem 5. *Let τ be a positive integer, and let $\mathbf{r}$ be an admissible factor of length τ where $\mathbf{r} = (r_0, \ldots, r_{\tau-1})$. Let $n_0^{(v)}$ be the initial iterate decided by $\left[\mathbf{r}^{(v)}\right]^\infty$ where*

$$n_0^{(v)} = \frac{\sum_{0 \le w < \tau} 3^w 2^{E_\tau^{(v)} - E_{w+1}^{(v)}}}{2^{E_\tau^{(v)}} - 3^\tau}.$$

We can write

$$n_0^{(v)} = 2^{e_0^{(v+\tau-1)}} k_0^{(v+\tau-1)} + 1,$$

and

$$k_u^{(v+\tau-u)} = 2^{e_0^{(v+\tau-1-u)}} k_{u+1}^{(v+\tau-1-u)} + b_u^{(v+\tau-1-u)}$$

for $u \ge 0$.

PROOF.

Assume the hypotheses and notation in the statement of the theorem.

As before, let D denote the non-zero quantity $2^{E_\tau^{(v)}} - 3^\tau$. We will make use of the following identity: for all $c \in \mathbb{Z}$, and for all integers $w \geq 0$, we have

$$E_\tau^{(c)} - E_{w+1}^{(c)} + e_0^{(c)} = E_\tau^{(c+1)} - E_w^{(c+1)} = E_\tau^{(c+1)} - E_{w+1}^{(c+1)} + e_w^{(c+1)}.$$

We will begin by demonstrating the equality

$$D\left(2^{e_0^{(v+\tau-1)}} k_0^{(v+\tau-1)} + 1\right) = D\left(3k_0^{(v)} + r_0^{(v)}\right);$$

dividing both sides of this equality by D yields the first desired equality in the theorem statement.

The following chain of equalities hold:

$$\begin{aligned}
D\left(2^{e_0^{(v+\tau-1)}} k_0^{(v+\tau-1)} + 1\right) &= \sum_{0 \leq w < \tau} 3^w 2^{E_\tau^{(v+\tau-1)} - E_{w+1}^{(v+\tau-1)} + e_0^{(v+\tau-1)}} s_0^{(v+w+\tau-1)} + 2^{E_\tau^{(v)}} - 3^\tau \\
&= \sum_{0 \leq w < \tau} 3^w 2^{E_\tau^{(v)} - E_w^{(v)}} \left[r_w^{(v+\tau)} - 1\right] + 2^{E_\tau^{(v)}} - 3^\tau \\
&= \sum_{0 \leq w < \tau} 3^w 2^{E_\tau^{(v)} - E_w^{(v)}} r_w^{(v)} - \sum_{1 \leq w < \tau+1} 3^w 2^{E_\tau^{(v)} - E_w^{(v)}} \\
&= 2^{E_\tau^{(v)}} r_0^{(v)} - 3^\tau r_\tau^{(v)} + \sum_{1 \leq w < \tau+1} 3^w 2^{E_\tau^{(v)} - E_w^{(v)}} \left[r_w^{(v)} - 1\right] \\
&= 3 \sum_{0 \leq w < \tau} 3^w 2^{E_\tau^{(v)} - E_{w+1}^{(v)}} \left[r_1^{(v+w)} - 1\right] + \left(2^{E_\tau^{(v)}} - 3^\tau\right) r_0^{(v)} \\
&= D\left(3k_0^{(v)} + r_0^{(v)}\right),
\end{aligned}$$

as desired.

We will now demonstrate the remainder of the claim. Let $u \geq 0$, and let $T_u^{(v)}$ denote the quantity $2^{e_0^{(v+\tau-u-1)}} k_{u+1}^{(v+\tau-u-1)} + b_u^{(v+\tau-u-1)}$. The next step of the proof makes use of the identity (3) from the previous proof.

We will write

$$\begin{aligned}
DT_u^{(v)} &= \sum_{0\le w<\tau} 3^w 2^{E_\tau^{(v+\tau-u-1)} - E_w^{(v+\tau-u-1)} + e_0^{(v+\tau-u-1)}} s_{u+1}^{(v+w+\tau-u-1)} + \left(2^{E_\tau^{(v)}} - 3^\tau\right) b_u^{(v+\tau-u-1)} \\
&= \sum_{0\le w<\tau} 3^w 2^{E_\tau^{(v-u)} - E_w^{(v-u)}} \left[d_u^{(v+w-u)} - b_u^{(v+w-u-1)}\right] + \left(2^{E_\tau^{(v)}} - 3^\tau\right) b_u^{(v-u-1)} \\
&= \sum_{0\le w<\tau} 3^w 2^{E_\tau^{(v-u)} - E_w^{(v-u)}} d_u^{(v+w-u)} - \sum_{1\le w<\tau+1} 3^w 2^{E_\tau^{(v-u)} - E_w^{(v-u)}} b_u^{(v+w-u-1)} \\
&= \sum_{0\le w<\tau} 3^w 2^{E_\tau^{(v-u)} - E_{w+1}^{(v-u)}} \left(2^{e_0^{(v+w-u)}} d_u^{(v+w-u)}\right) - \sum_{0\le w<\tau} 3^w 2^{E_\tau^{(v-u)} - E_{w+1}^{(v-u)}} \left(3b_u^{(v+w-u)}\right) \\
&= \sum_{0\le w<\tau} 3^w 2^{E_\tau^{(v+\tau-u)} - E_{w+1}^{(v+\tau-u)}} s_u^{(v+w+\tau-u)} \\
&= Dk_u^{(v+\tau-u)},
\end{aligned}$$

whereby completing the argument.

□

3.4 An Illustrated Example

We will compute the first several terms of the 3-adic and **2**-adic expansions of the quotient generated by $\mathbf{r}^\infty$ where $\mathbf{r} = \mathbf{1221}$.

As per Böhm and Sontacchi, the initial iterate value n_0 equals

$$\frac{3^0 2^{1+1+2} + 3^1 2^{1+2} + 3^2 2^2 + 3^3 2^0}{2^{2+1+1+2} - 3^4} = -\frac{103}{17}.$$

We can write $n_0 = 3k_0 + 1$, where

$$k_0 = \frac{3^0 2^{1+1+2}[2-1] + 3^1 2^{1+2}[2-1] + 3^2 2^2 [1-1] + 3^3 2^0 [1-1]}{2^{2+1+1+2} - 3^4} = -\frac{40}{17};$$

the summands in the numerator are scaled by the entries in the s_0 column of Table 2a.

The 3-adic expansion of k_0 begins with the digits $110112\cdots$, and its **2**-adic expansion (with gradation factor **2112**) begins with the digits $001110\cdots$. The connection between these digits and the shifts of the quotient is illustrated in Table 2.

One can derive the canonical expansions and shifts of the quotients $k_0^{(1)}$, $k_0^{(2)}$, and $k_0^{(3)}$ from the same table.

$\mathcal{O}$ (**1221**)	s_0	s_1	s_2	s_3	s_4	s_5
1221	1	1	0	1	1	-1
2211	1	-1	-1	1	0	1
2112	0	0	1	-1	-1	0
1122	0	1	0	0	1	0

(a) The term $s_u^{(c)} = d_{u-1}^{(c+1)} - b_{u-1}^{(c)}$, where $d_{-1}^{(c)} :=$ $r_c - 1$ and $b_{-1}^{(c)} := 0$.

$\mathcal{O}$ (**1221**)	s_0	s_1	s_2	s_3	s_4	s_5
1221	1	1	0	1	1	2
2211	1	-1	-1	1	0	1
2112	0	0	2	1	1	0
1122	0	1	0	0	1	0

$\mathcal{O}$ (**1221**)	s_0	s_1	s_2	s_3	s_4	s_5
1221	1	1	0	1	1	3
2211	1	-1	1	1	0	1
2112	0	0	1	1	-1	0
1122	0	1	0	0	1	0

(b) The 3-adic and **2**-adic expansions of the quotient generated by $\mathbf{1221}^\infty$ derived from the array of $s_u^{(c)}$ terms. The 3-adic digit $d_u^{(c)} \underset{3}{\equiv} 2^{e_c} s_u^{(c)}$, and the **2**-adic digit $b_u^{(c)} \underset{2^{e_c}}{\equiv} s_u^{(c)}$.

Table 2

An illustrated example of Theorem 3 where $\mathbf{r} = \mathbf{1221}$. The u-th 3-adic shift of k_0 is evaluated by scaling the mixed, prime-power summands in the numerator by the s_u column of the array in Table 2a.

4 Conclusions

This article presents a novel focus on the p-adic properties ($p \in \{2, 3\}$) of the quotients of a periodic iterate trajectory in the $3x+1$ Accelerated First-Inverse map. We express the u-th interate of the periodic trajectory as

$$n_u = 3\left(2^{E_u} k_u + P_u\right) + r_u,$$

and we analyzed the trajectories of the various terms within this formulation. We introduced the ring of mixed-radix **m**-adic integers, and we derived a recurrence on the set of 3-adic and **2**-adic digits for computing both the canonical expansions and the adic shifts of the quotients of the iterates.

Some future directions for this effort are summarized here.

(1) *Cantor meets Collatz*: We view the function $\mathcal{B} : Z_3 \to Z_3$ as a mapping from the 3-adics (over the alphabet $\{1, 2\}$) to the 3-adics. What if the prescribed iterate remainder sequence is pre-periodic? Aperiodic?
(2) *Generalized Collatz Systems*: Let m and l be coprime integers where $m > l \geq 2$, and let τ be a positive integer. Let $\mathbf{e}$ and $\mathbf{f}$ denote factors of positive

integers, and let $(E_u)_{u\geq 0}$ and $(F_u)_{u\geq 0}$ denote the sequences of prefix sums of $\mathbf{e}$ and $\mathbf{f}$, respectively. Can we apply the analyses herein to compute the $\mathbf{m}$-adic and $\mathbf{l}$-adic expansions and shifts of

$$\frac{\sum_{0\leq w<\tau} m^{F_w} l^{E_\tau - E_{w+1}}}{l^{E_\tau} - m^{F_\tau}}?$$

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